

# BETTI-LINEAR IDEALS

DANIEL WOOD

**ABSTRACT.** We introduce the notion of a Betti-linear monomial ideal, which generalizes the notion of lattice-linear monomial ideal introduced by Clark. We provide a characterization of Betti-linearity in terms of Tchernev's poset construction. As an application we obtain an explicit canonical construction for the minimal free resolutions of monomial ideals having pure resolutions.

## INTRODUCTION

Understanding the structure of minimal free resolutions of monomial ideals is an active area of research in commutative algebra. One important aspect of this problem is investigating what it means for the resolution to be linear or close to linear. For example, Eagon and Reiner [ER] have shown that an ideal  $I$  with linear resolution has an Alexander dual that is Cohen-Macaulay; Herzog and Hibi [HH] introduce and study the notion of componentwise linear ideals; and Clark [Cl] introduced the notion of lattice-linearity and proved a criterion for  $I$  to be lattice-linear in terms of acyclicity properties of the so-called *poset construction*.

In this paper we introduce the new notion of a *Betti-linear* monomial ideal that generalizes the notion of lattice-linearity. We provide a criterion for Betti-linearity in terms of the poset construction. A class of Betti-linear monomial ideals are those having pure resolution. They arise in connection with Boij-Soderberg theory [BS, BS2, ES, EFW], and were studied recently by Francisco, Mermin, and Schweig [FMS]. Thus, our main result yields an explicit description of the structure of the minimal free resolution of  $I$  when  $I$  has pure resolution.

To be specific, let  $B$  be the Betti poset [CM, TV] of the monomial ideal  $I$  with minimal free resolution  $\mathcal{F} = (F_k, \phi_k)$ . Thus,  $B$  is the set of monomial degrees of the basis elements of  $\mathcal{F}$  ordered by divisibility. We say that the ideal  $I$  is *Betti-linear* if we can fix homogeneous bases  $B_k$  of the free modules  $F_k$  for all  $k$  so that for any  $i \geq 1$  and any  $\tau \in B_i$ , the element

$$\phi_i(\tau) = \sum_{\gamma \in B_{i-1}} [\tau : \gamma] \cdot \gamma$$

has the property that if  $[\tau, \gamma] \neq 0$  then the monomial degree of  $\gamma$  is covered in the poset  $B$  by that of  $\tau$ . In our main result, Theorem 2.5, we show that  $I$  is Betti-linear if and only if  $\mathcal{F}$  can be recovered from the poset construction applied to the Betti poset of the ideal  $I$ .

The structure of this paper is as follows: In Section 2, we introduce the notion of Betti linearity, examine a few examples, and state our main theorem. In Section 3 we discuss the poset construction in detail. Section 4 provides a few key properties of the Betti poset. Finally, in Section 5 we give a proof of our main theorem.

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## 1. PRELIMINARIES

Throughout this paper,  $k$  is a fixed field and  $R = k[x_1, \dots, x_n]$  is a polynomial ring over  $k$ . The ring  $R$  as a vector space over  $k$  is the direct sum

$$R = \bigoplus_{\alpha \in \mathbb{Z}^n} R_\alpha$$

where  $R_\alpha = k \cdot x^\alpha$  for  $\alpha \in \mathbb{N}^n$  and is 0 otherwise. Thus,  $R$  is a  $\mathbb{Z}^n$ -graded or *multigraded*  $k$ -algebra. Let  $\mathfrak{m} = (x_1, \dots, x_n)$  be the unique graded maximal ideal of the ring  $R$ . For  $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ , we write

$$x^a = x_1^{a_1} \cdots x_n^{a_n} \in R.$$

and we will always identify  $x^a$  with its exponent  $a \in \mathbb{Z}^n$ . For  $\gamma \in \mathbb{Z}^n$  we write  $R(-\gamma)$  for the shifted multigraded  $R$ -module with  $R(-\gamma)_\alpha = R_{\alpha-\gamma}$ . Thus,  $R(-\gamma)$  is free of rank 1 with basis a single homogeneous element of degree  $\gamma$ .

Let  $(P, \leq)$  be a poset. Let  $\sigma \subseteq P$  be a subset of  $P$ . If the meet or join of  $\sigma$  exist, they are denoted as  $\wedge \sigma$  and  $\vee \sigma$  respectively. If  $\sigma$  has the form

$$x_0 < x_1 < \dots < x_k$$

then  $\sigma$  is called a *chain of length  $k$*  or a  *$k$ -chain* of  $P$ . For any element  $x \in P$ , we define the *dimension* of  $x$  to be

$$d_P(x) = d(x) = \sup \{k : x_0 < \dots < x_k = x\}$$

Any subset of  $P$  that is comprised of elements that are pairwise incomparable is called an *anti-chain*. An element  $y \in P$  is said to be *covered* by  $x$ , which we denote  $y \lessdot x$ , when it is true that  $y < x$  and there exists no  $z \in P$  so that  $y < z < x$ . Denote by  $P_{<x}$  the subset of  $P$  given by

$$P_{<x} = \{z \in P : z < x\},$$

with  $P_{\leq x}$  defined analogously. Let  $A$  be the set of minimal elements of  $P$ , and note that  $a \in A$  if and only if  $d(a) = 0$ .

To a poset  $P$ , we associate its order complex  $\Delta(P)$  which is an abstract simplicial complex whose vertices are the elements of  $P$  and for each  $k > 0$ , the  $k$ -dimensional faces of  $\Delta(P)$  are the  $k$ -chains of  $P$ . When we refer to the topological properties of the poset  $P$ , we are referring to the topological properties of the abstract simplicial complex  $\Delta(P)$ .

Conversely, given a simplicial complex  $S$ , one may define the face poset of  $S$ ,  $F(S)$ , which is the set of nonempty faces of  $S$  partially ordered by inclusion. Under these correspondences, we can identify the first barycentric subdivision of  $S$  as  $\mathbf{sd}(S) = \Delta(F(S))$ .

## 2. BETTI-LINEARITY

Let  $I \subseteq R$  be a monomial ideal with minimal  $\mathbb{Z}^n$ -graded free resolution

$$\mathcal{F} : 0 \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \dots \xleftarrow{\phi_{i-1}} F_{i-1} \xleftarrow{\phi_i} F_i \xleftarrow{\phi_{i+1}} \dots \xleftarrow{\phi_m} F_m \longleftarrow 0.$$

For each  $i$ , write

$$mdeg : B_i \longrightarrow \mathbb{Z}^n$$

for the map that assigns to a basis element  $\tau \in B_i$  its  $\mathbb{Z}^n$ -degree. In particular,  $mdeg(\tau)$  is an element of the Betti poset  $B$  of  $I$  over  $k$ . We also write  $deg(\tau)$  for the total degree of  $\tau$  in  $\mathbb{Z}$ .

**Definition 2.1.** The ideal  $I$  is called *Betti-linear* if in the minimal free resolution  $\mathcal{F}$  we can fix a  $\mathbb{Z}^n$ -graded basis  $B_t$  of  $F_t$  for each  $t$  so that for all  $i \geq 1$ , and for all  $\tau \in B_i$ , we have that

$$\phi_i(\tau) = \sum_{\gamma \in B_{i-1}} [\tau : \gamma] \gamma$$

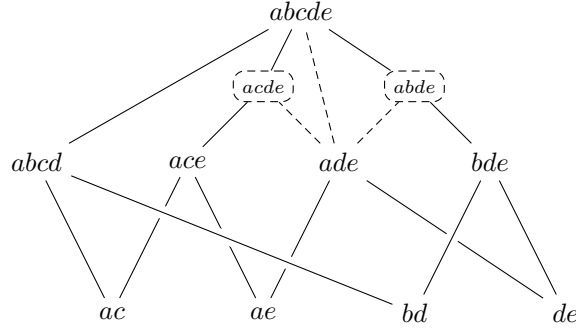
has that property that if  $[\tau : \gamma] \neq 0$  then  $mdeg(\gamma) \leq_B mdeg(\tau)$ .

An important invariant of  $I$  is its *lcm-lattice* consisting of the joins in  $\mathbb{N}^n$  of subsets of the degrees of the minimal generators of  $I$ . We denote by  $L$  the lcm-lattice of  $I$  without its minimal element. It is well known that the Betti poset is a subposet of  $L$ , thus the notion of Betti-linearity generalizes the previously established notion of lattice-linearity due to Clark [Cl].

**Example 2.2.** Let  $k$  be any field. Let  $R = k[a, b, c, d, e]$  be the polynomial ring over  $k$  in 5 variables. Consider the ideal  $I = (ac, bd, ae, de) \subseteq R$ . The minimal free resolution of this ideal takes the form

$$0 \longleftarrow R \longleftarrow R^4 \longleftarrow R^4 \longleftarrow R \longleftarrow 0$$

and is known to be Betti-linear, however, the ideal  $I$  does not have a lattice-linear resolution. For the given ideal  $I$ , the lcm-lattice of  $I$  is



Here, the dashed entries indicate elements  $\alpha$  for which  $\alpha \notin B$ .

**Example 2.3.** Let  $k$  be any field. Let  $R = k[a, b, c, d]$  be a polynomial ring over  $k$ . Consider the ideal

$$I = (ad, bc^2, c^3d^4, a^2, ab, ac, b^2c, c^4d^3, b^3, c^5d^2, cd^6, c^7).$$

This is a stable ideal of  $R$  with respect to the ordering  $a < b < c < d$ . The formulas of Eliahou and Kervaire [EK] show that in homological degree 1 of the minimal resolution of  $I$  there is a single direct summand  $R(-(1, 1, 2, 0))$ , and provide a generator  $\tau$  for it. They also show that in the same homological degree there is a unique (up to a constant multiple) basis element  $\sigma$  with

$$mdeg(\sigma) = (1, 1, 1, 0);$$

and that all other basis elements degrees are not comparable with the degree of  $\tau$ . Furthermore,  $\tau$  maps onto an element with a component that is a non-zero multiple of the unique basis element  $\eta$  in homological degree 0 with  $mdeg(\eta) = (1, 1, 0, 0)$ .

Clearly, one has that  $\eta < \sigma < \tau$  in  $B$ , so that  $\tau$  is not a cover for  $\eta$ . Since any other choice of a homogeneous basis element of degree  $(1, 1, 2, 0)$  in homological degree 1 has to be of the form  $r\tau + s\sigma$  for some constants  $r, s \in k$  with  $r \neq 0$ , it is immediate that any such choice will violate the Betti-linearity condition as well. It follows that the stable ideal  $I$  is not Betti-linear.

Next, we show that monomial ideals with pure resolution are in fact Betti-linear ideals:

**Proposition 2.4.** *Let  $I \subseteq R = k[x_1, \dots, x_n]$  be a monomial ideal with pure resolution. Then  $I$  is a Betti-linear ideal.*

*Proof.* Suppose that the minimal free resolution of  $I$  is

$$\mathcal{F} : 0 \longleftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \dots \xleftarrow{\partial_{i-1}} F_{i-1} \xleftarrow{\partial_i} F_i \xleftarrow{\partial_{i+1}} \dots \xleftarrow{\partial_k} F_k \longleftarrow 0$$

then because this resolution is pure, it must be the case that for each basis element  $\sigma \in F_i$  that  $\deg(\sigma) = d_i$ . Suppose now that  $\sigma \in F_i$  and  $\tau \in F_{i-1}$  are basis elements such that  $[\sigma : \tau] \neq 0$ . Clearly,  $\tau < \sigma$ . We want to show that  $\tau \leq \sigma$ .

Suppose that there was  $\gamma \in B$  such that  $\tau < \gamma < \sigma$ . The resolution is minimal, so therefore  $\deg(\tau) = d_{i-1} < \deg(\gamma) < d_i = \deg(\sigma)$ . This is impossible, as the elements of  $B$  must have total degree which is an element of the set  $\{d_0, d_1, \dots, d_{k-1}, d_k\}$ , a strictly increasing sequence.

It follows that no such  $\gamma$  exists and that  $\tau \leq \sigma \in B$ .  $\square$

The main theorem of this paper provides a characterization for Betti-linear ideals that generalizes [Cl, Theorem 3.3]:

**Theorem 2.5.** *A monomial ideal  $I \subseteq R$  is Betti-linear if and only if the poset construction on  $B$  recovers the minimal free resolution of the ideal  $I$ . In particular, this provides an explicit canonical construction of the minimal free resolution of monomial ideals with pure resolution.*

We postpone the proof of Theorem 2.5 until Section 5.

### 3. THE POSET CONSTRUCTION

Let  $P$  be a poset. For  $\alpha \in P$ , let  $\Delta_\alpha = \Delta(P_{<\alpha})$  and  $\Delta_{\leq\alpha} = \Delta(P_{\leq\alpha})$ . Thus

$$\Delta_\alpha = \bigcup_{\lambda < \alpha} \Delta_{\leq\lambda}.$$

Fix  $\lambda < \alpha$ , and set

$$\Delta_{\alpha,\lambda} = \Delta_{\leq\lambda} \cap \left( \bigcup_{\lambda \neq \beta < \alpha} \Delta_{\leq\beta} \right).$$

**Definition 3.1.** [Cl] For  $i \geq 0$ , set  $\mathcal{D}_{i,\alpha} = \tilde{H}_{i-1}(\Delta_\alpha, k)$  and set

$$\mathcal{D}_i = \bigoplus_{\alpha \in P} \mathcal{D}_{i,\alpha}$$

**Remark 3.2.** If  $i = 0$  and  $\alpha \in A$ , then  $\Delta_\alpha$  is the empty simplicial complex, and so we see that

$$\mathcal{D}_{0,\alpha} = \tilde{H}_{-1}(\{\emptyset\}, k)$$

a 1-dimensional  $k$ -vector space. On the other hand, if  $i = 1$  and  $\alpha \notin A$ , then  $\Delta_\alpha \neq \{\emptyset\}$ . From this it follows that  $\mathcal{D}_{0,\alpha} = \tilde{H}_{-1}(\Delta_\alpha, k) = 0$ . Thus,

$$\mathcal{D}_0 = \bigoplus_{\alpha \in A} \mathcal{D}_{0,\alpha} = \bigoplus_{\alpha \in A} \tilde{H}_{-1}(\{\emptyset\}, k) \cong \bigoplus_{\alpha \in A} k.$$

Given  $\lambda < \alpha \in P$ , consider the Mayer-Vietoris sequence in reduced homology for the triple

$$(\Delta_{\leq \lambda}, \bigcup_{\lambda \neq \beta < \alpha} \Delta_{\leq \beta}, \Delta_\alpha)$$

Write  $j : \tilde{H}_{i-2}(\Delta_{\alpha,\lambda}, k) \rightarrow \tilde{H}_{i-2}(\Delta_\lambda, k)$  for the map induced in homology by the inclusion map and let

$$\partial_{i-1}^{\alpha,\lambda} : \tilde{H}_{i-1}(\Delta_\alpha, k) \rightarrow \tilde{H}_{i-2}(\Delta_{\alpha,\lambda}, k)$$

be the connecting homomorphism from the Mayer-Vietoris sequence. Recall that for  $[c] \in \tilde{H}_{i-2}(\Delta_\alpha, k)$ , this homomorphism is given by

$$\partial_{i-1}^{\alpha,\lambda}([c]) = [d_{i-1}(c')] \in \tilde{H}_{i-2}(\Delta_{\alpha,\lambda}, k)$$

where we have that  $c' + c'' = c \in \tilde{C}_{i-2}(\Delta_\alpha, k)$  and  $c'$  and  $c''$  are any components of  $c$  that are supported by  $\Delta_{\leq \lambda}$  and  $\bigcup_{\lambda \neq \beta < \alpha} \Delta_{\leq \beta}$  respectively, and  $d$  is the usual simplicial boundary map.

**Definition 3.3.** [Cl] For  $i \geq 1$ , define  $\phi_i : \mathcal{D}_i \rightarrow \mathcal{D}_{i-1}$  componentwise by

$$\phi_i|_{\mathcal{D}_{i,\alpha}} = \sum_{\lambda < \alpha} \phi_i^{\alpha,\lambda}$$

where the map

$$\phi_i^{\alpha,\lambda} : \mathcal{D}_{i,\alpha} \rightarrow \mathcal{D}_{i-1,\lambda}$$

is the composition  $\phi_i^{\alpha,\lambda} = j \circ \partial_{i-1}^{\alpha,\lambda}$ . We define  $\mathcal{D}(P, k)$  as the sequence of modules and maps

$$\mathcal{D}(P, k) : \mathcal{D}_0 \xleftarrow{\phi_1} \mathcal{D}_1 \leftarrow \dots \mathcal{D}_{i-1} \xleftarrow{\phi_i} \mathcal{D}_i \dots \leftarrow \mathcal{D}_k \leftarrow 0,$$

and we refer to  $\mathcal{D}(P, k)$  as the *poset construction* on  $P$  over  $k$ .

Notice that the vector space maps  $\phi_i$  are canonical and determined by the structure of the homology of the filters  $P_{\leq \alpha}$  in the poset  $P$ .

Suppose that  $\eta : P \rightarrow \mathbb{Z}^n$  is a map of partially ordered sets and  $A$  is the set of minimal elements of the poset  $P$ . Let  $N \subseteq R$  be the ideal minimally generated by the set

$$\{x^{\eta(a)} : a \in A\}.$$

Then the sequence  $\mathcal{D}(P, k)$  can be homogenized using the map  $\eta$  to produce a sequence of multigraded  $R$ -modules and  $R$ -module homomorphisms which will approximate a free resolution of the multigraded module  $R/N$ .

We homogenize in the following way: For  $i \geq 0$ , set

$$\mathcal{F}_i(\eta) = \bigoplus_{\lambda \in P} \mathcal{F}_{i,\lambda}(\eta) = \bigoplus_{\lambda \in P} \mathcal{D}_{i,\lambda} \otimes_k R(-\eta(\lambda))$$

and note the multigrading satisfies  $mdeg(v \otimes x^a) = a + \eta(\lambda)$  for each  $v \in \mathcal{D}_{i,\lambda}$ .

The differential in this sequence is defined componentwise in homological degree  $i \geq 1$  by the map  $\partial_i : \mathcal{F}_i(\eta) \rightarrow \mathcal{F}_{i-1}(\eta)$  given by

$$\partial_i|_{\mathcal{F}_{i,\alpha}(\eta)} = \sum_{\lambda \leq \alpha} \partial_i^{\alpha,\lambda},$$

where  $\partial_i^{\alpha,\lambda} : \mathcal{F}_{i,\alpha}(\eta) \rightarrow \mathcal{F}_{i-1,\lambda}(\eta)$  has the form  $\partial_i^{\alpha,\lambda} = x^{\eta(\alpha)-\eta(\lambda)} \otimes \phi_i^{\alpha,\lambda}$  for  $\lambda \leq \alpha$ . This gives a sequence of multigraded  $R$ -modules and maps

$$\mathcal{F}(\eta) : \dots \rightarrow \mathcal{F}_i(\eta) \xrightarrow{\partial_i} \mathcal{F}_{i-1}(\eta) \rightarrow \dots \rightarrow \mathcal{F}_1(\eta) \xrightarrow{\partial_1} \mathcal{F}_0(\eta).$$

**Definition 3.4.** [Cl] If  $\mathcal{F}(\eta)$  is an acyclic complex of multigraded modules, it is called a *poset resolution* of the ideal  $N$ .

#### 4. PROPERTIES OF BETTI POSETS

Let  $I$  be a monomial ideal of  $R$ . Recall that  $L$  denotes the lcm-lattice of  $I$  without its minimal element, and denote the inclusion of the Betti poset  $B$  into  $L$  by  $\iota : B \rightarrow L$ . The formulas of Gasharov, Peeva, and Welker [GPW] tell us that  $\alpha \in L - B$  if and only if  $\tilde{H}_*(\Delta(L_{<\alpha}), k) = 0$ . By Tchernev and Varisco [TV] and Clark and Mapes [CM], the induced map on homology  $\iota_* : H_*(\Delta(B), k) \rightarrow H_*(\Delta(L), k)$  is an isomorphism.

To analyze the map  $\iota$  better, we consider the following simplicial complex associated to any poset  $P$ . Recall that  $A$  is the set of minimal elements of  $P$ .

**Definition 4.1.** (a) We define  $\Theta(P)$  to be the abstract simplicial complex on the set of vertices  $A$  with faces the collection of all  $F \subseteq A$  so that  $F$  is bounded in  $P$ .

(b) When  $\rho : P \rightarrow Q$  is a morphism of posets sending minimal elements of  $P$  to minimal elements of  $Q$ , we write  $\Theta(\rho)$  for the induced map of simplicial complexes  $\Theta(P) \rightarrow \Theta(Q)$ .

We briefly discuss the connection between  $\Theta(P)$  and crosscut complexes. Recall that a subset  $C \subseteq P$  of a poset  $P$  is called a *crosscut* if (1)  $C$  is an antichain, (2) for every finite chain  $\sigma$  in  $P$  there exists some element in  $C$  which is comparable to every element in  $\sigma$ , and (3) if  $E \subseteq C$  is bounded, then the join  $\vee E$  or the meet  $\wedge E$  exist in  $P$ . Here, when we say  $E$  is bounded, we mean that  $E$  has an upper or a lower bound in the poset  $P$ .

If  $C \subseteq P$  is a crosscut, then the crosscut complex  $\Gamma(P, C)$  is defined to be the simplicial complex consisting of the bounded subsets of  $C$ . It is well known that  $\Gamma(P, C)$  is homotopy equivalent to  $\Delta(P)$ .

Now observe that if  $A$  forms a crosscut of  $P$ , then  $\Gamma(P, A) = \Theta(P)$ . In particular, since  $A$  is a crosscut of  $L$ , we obtain that  $\Gamma(L, A) = \Theta(L)$  and hence is homotopy equivalent to  $\Delta(L)$ .

Next, consider the map  $\Theta(\iota) : \Theta(B) \rightarrow \Theta(L)$ . We would like to show that the map  $\Theta(\iota)_* : \tilde{H}_i(\Theta(B_{<\alpha}), k) \rightarrow \tilde{H}_i(\Theta(L_{<\alpha}), k)$  is an isomorphism for all  $i$ . To do so, we will make use of the following key lemma:

**Lemma 4.2.** *The relative homology groups  $\tilde{H}_i(\Theta(L_{<\alpha}), \Theta(B_{<\alpha}), k)$  are 0 for all  $i$ .*

*Proof.* Let  $c \in C_i(\Theta(L_{<\alpha}), k)$  be an  $i$ -chain such that  $\partial_i c \in \Theta(B_{<\alpha})$ . Then it is enough to show that there exists  $b \in \Theta(L_{<\alpha})$  so that  $c - \partial_n b \in \Theta(B_{<\alpha})$ .

We know that  $c$  is an element of  $C_i(\Theta(L_{<\alpha}), k)$ , so we may write

$$c = \sum_{F \in \Theta(L_{<\alpha})} a_F F,$$

and without loss of generality, we may assume that each  $F$  is not in  $\Theta(B_{<\alpha})$ . For each  $F$ , consider  $\alpha_F = \bigvee_L \{a : a \in F\}$ , hence  $\{\alpha_1, \dots, \alpha_k\}$  are joins of the faces  $F$  and  $\alpha_j \notin B$ . Let  $\alpha_1, \dots, \alpha_p$  be the maximal elements among all the  $\alpha_j$ . It is necessarily true that there is no  $\beta \in B$  so that  $\alpha_j < \beta < \alpha$  in  $L$  for any  $j$ .

For each  $j$ , let  $z_{\alpha_j} = \sum_{\bigvee(F)=\alpha_j} a_F F$ . Then of course we have that  $c = z_{\alpha_1} + \dots + z_{\alpha_k}$ , hence,

$$\partial_i z_{\alpha_1} = \partial_i c - \sum_{2 \leq j \leq k} \partial_i z_{\alpha_j}$$

For a chain  $\sigma$  with  $\partial\sigma = \sum c_m \tau_m$ , define  $\text{supp}(\partial\sigma) = \{\tau_m : c_m \neq 0\}$ . Now suppose that  $F$  is a face of  $\text{supp}(\partial_i z_{\alpha_1})$  such that  $\alpha_F = \alpha_1$ . Then notice that  $F$  is not in the support of  $\sum_{2 \leq j \leq k} \partial_i z_{\alpha_j}$ , so therefore we must have that  $F \in \text{supp}(\partial_i c)$ . This however implies that  $F \in \Theta(B_{<\alpha})$ , which implies there is  $\beta \in B$  so that  $\alpha_1 < \beta < \alpha$ . Since this is a contradiction, we must conclude that each face  $F \in \text{supp}(\partial_i z_{\alpha_1})$  satisfies  $\alpha_F < \alpha_1$ .

From this we see that  $\partial_i z_{\alpha_1} \in \Theta(L_{<\alpha_n})$  while  $z_{\alpha_1} \in \Theta(L_{\leq \alpha_1})$ . Therefore,  $z_{\alpha_1}$  is a relative cycle of the pair  $(\Theta(L_{\leq \alpha_1}), \Theta(L_{<\alpha_1}))$ . However, the relative chain complex of the pair  $(\Theta(L_{\leq \alpha_1}), \Theta(L_{<\alpha_1}))$  is acyclic over  $k$  as  $\Theta(L_{\leq \alpha})$  is a simplex and  $\Theta(L_{<\alpha_1})$  is acyclic over  $k$  as it is homotopy equal to  $\Gamma(L_{<\alpha_1})$  and  $\alpha_n \notin B$ . So we get that  $z_{\alpha_1}$  is also a relative boundary. Due to this, we can find  $b_1 \in \Theta(L_{\leq \alpha})$  so that  $z_{\alpha_1} - \partial_i b_1 \in \Theta(L_{<\alpha})$ .

Repeat this for  $\alpha_2, \dots, \alpha_p$  to get  $b_2, \dots, b_p$ . Then consider the element

$$c_1 := c - \sum_{1 \leq j \leq p} \partial_i b_j.$$

The element  $c_1$  may not yet satisfy the conditions of the lemma, but for certain, if  $G$  is a face appearing in the expression for  $c_1$ , we have that  $\alpha_G < \alpha_j$  for (at least) one of the  $\alpha_j, j = 1, \dots, p$ .

If  $c_1$  does not satisfy the lemma, we may iterate the same procedure. Letting  $l = \max\{d(\alpha_1), \dots, d(\alpha_p)\}$ , we see that after obtaining  $c_l$ , the process has terminated with  $c_l \in \Theta(B_{<\alpha})$  as desired.  $\square$

**Proposition 4.3.** *The map  $\Theta(j)_* : \tilde{H}_i(\Theta(B_{<\alpha}), k) \longrightarrow \tilde{H}_i(\Theta(L_{<\alpha}), k)$  is an isomorphism for all  $i$ .*

*Proof.* Consider the long exact sequence in homology

$$\begin{aligned} \dots \longrightarrow \tilde{H}_{i+1}(\Theta(L_{<\alpha}), \Theta(B_{<\alpha}), k) &\longrightarrow \tilde{H}_i(\Theta(B_{<\alpha}), k) \longrightarrow \tilde{H}_i(\Theta(L_{<\alpha}), k) \\ &\longrightarrow \tilde{H}_i(\Theta(L_{<\alpha}), \Theta(B_{<\alpha}), k) \longrightarrow \dots \end{aligned}$$

and apply the lemma above.  $\square$

Consider the map  $\Psi : \Delta(L) \longrightarrow \Theta(L)$  given by

$$\sigma \mapsto \{x \in A : \sigma \in \Delta(L_{\geq x})\}$$

where  $L_{\geq x} = \{y \in L : y \geq x\}$ . Note that  $\Psi$  is precisely the map that realizes the homotopy equivalence between  $\Delta(L)$  and  $\Theta(L)$ , see [Bj]. Restricting  $\Psi$  to  $B$ , we

obtain the map  $\psi : \Delta(B) \longrightarrow \Theta(B)$  given by

$$\sigma \mapsto \{x \in A : \sigma \in \Delta(B_{\geq x})\}.$$

While  $\Psi$  is always a homotopy equivalence, in general it will not be the case that  $\psi$  is a homotopy equivalence. However we can show that it is an isomorphism in homology. For this we use the following lemma:

**Lemma 4.4.** *The diagram*

$$(4.5) \quad \begin{array}{ccc} C(\Delta(B), k) & \xrightarrow{\Delta(j)} & C(\Delta(L), k) \\ sd \downarrow & & \downarrow sd \\ C(sd(\Delta(B)), k) & \xrightarrow{sd(\Delta(j))} & C(sd(\Delta(L)), k) \\ sd(\psi) \downarrow & & \downarrow sd(\Psi) \\ C(sd(\Theta(B)), k) & \xrightarrow{sd(\Theta(j))} & C(sd(\Theta(L)), k) \\ sd \uparrow & & \uparrow sd \\ C(\Theta(B), k) & \xrightarrow{\Theta(j)} & C(\Theta(L), k) \end{array}$$

is commutative.

*Proof.* The top and bottom square commute for trivial reasons.

To show that the middle square is commutative, let  $\Psi : \Delta(L) \longrightarrow \Theta(L)$  and note that  $sd(\Psi)([\sigma_0 \subset \dots \subset \sigma_k]) = [\Psi(\sigma_k) \subset \dots \subset \Psi(\sigma_0)]$ , or 0 if the simplex defined by this formula is degenerate. Assume that each  $\sigma_i \in P(\Delta(B))$ . Then  $\Psi(\sigma_i) = \{a \in A : \sigma_i \in \Delta(L_{\geq a})\} = \{a \in A : \sigma_i \in \Delta(B_{\geq a})\}$ , the last equality being an equality of sets, as  $\sigma_i \in P(\Delta(B))$  tells us that if  $\sigma \in \Delta(L_{\geq a})$ , then we also have  $\sigma \in \Delta(B_{\geq a})$ . We conclude that  $sd(\psi)$  has image contained in  $sd(\Theta(B))$ .  $\square$

**Proposition 4.6.** *For any  $\alpha \in B$ , the map*

$$\tilde{H}(\psi) : \tilde{H}_i(\Delta(B_{<\alpha}), k) \longrightarrow \tilde{H}_i(\Theta(B_{<\alpha}), k)$$

is an isomorphism.

*Proof.* All maps involved in 4.5 are homology isomorphisms with the exception of the map  $sd(\psi) : sd(\Delta(B)) \longrightarrow sd(\Theta(B))$ . By the commutativity of the diagram,  $sd(\psi)$  is also a homology isomorphism, hence so is  $\psi$ .  $\square$

## 5. THE PROOF OF THE MAIN THEOREM

**Definition 5.1.** For  $\lambda \in B$ , let  $G_\lambda$  be the full simplex on the  $A_\lambda = \{a \in A : a \leq \lambda\}$ . Fix  $\alpha \in B$ . For each  $\lambda < \alpha$ , define  $\Theta_{\alpha, \lambda}$  to be the complex

$$\Theta_{\alpha, \lambda} = G_\lambda \bigcap \left( \bigcup_{\lambda \neq \beta <_B \alpha} G_\beta \right)$$

While it is not necessarily true that  $\Theta_{\alpha, \lambda} \subset \Theta(B_{<\lambda})$ , it is true that  $\Theta_{\alpha, \lambda} \subset \Theta(L_{<\lambda})$ . Denote this inclusion map by  $j$ . Further, we know that the inclusion  $\Theta(B_{<\lambda}) \longrightarrow \Theta(L_{<\lambda})$  is a homology isomorphism. Let  $g_*$  be the homology inverse of this map. We set

$$\mu_* = g_* \circ j_* : \tilde{H}_*(\Theta_{\alpha, \lambda}, k) \longrightarrow \tilde{H}_{*-1}(\Theta(B_{<\lambda}), k).$$



Let  $\partial_i^{\alpha,\lambda} : \tilde{H}_i(\Theta(B_{<\alpha}), k) \longrightarrow \tilde{H}_{i-1}(\Theta_{\alpha,\lambda}, k)$  be the connecting homomorphism in the Mayer-Vietoris sequence for the triple  $(G_\lambda, \bigcup_{\lambda \neq \beta \leq \alpha} G_\beta, \Theta(B_{<\alpha}))$ .

Consider any sequence  $\mathcal{S}$  of morphisms of free multigraded modules. This can be decomposed as

$$\mathcal{S} = \bigoplus_{\alpha \in \mathbb{Z}^n} S_\alpha$$

where each  $S_\alpha$  is a sequence of maps of vector spaces called the multigraded strand of  $\mathcal{S}$  in degree  $\alpha$ . Let  $(S_\alpha)_i$  denote the  $i$ th vector space. Notice also that

$$(\mathbf{m}\mathcal{S})_\alpha = \sum_{\beta < \alpha} x^{\alpha-\beta} S_\beta \subset S_\alpha.$$

We will identify  $x^{\alpha-\beta} S_\beta$  with  $S_\beta$  so one may write  $S_\beta \subset S_\alpha$  for  $\beta < \alpha$ .

Now we suppose that  $I$  is a monomial ideal over the polynomial ring  $R = k[x_1, \dots, x_m]$ . Suppose that  $\mathcal{F}$  is its minimal free resolution and for each  $i$ , let  $B_i$  be the chosen basis of  $F_i$  so that for each  $v \in B_i$ , we have that

$$\partial_i(v) = \sum_{t \in B_{i-1}} [v : t] \cdot t$$

is such that if  $[v : t] \neq 0$  then  $mdeg(t) <_B mdeg(v)$ . Let  $F_{i,\alpha}$  be the free submodule of  $F_i$  spanned by the set

$$B_{i,\alpha} = \{v \in B_i : mdeg(v) = \alpha\}$$

Then we have that

$$F_i = \bigoplus_{\alpha \in B} F_{i,\alpha}.$$

Using the notation

$$V_{i,\beta} = k \langle v : v \in B_{i,\beta} \rangle$$

we write

$$(F_\alpha)_i = \bigoplus_{\beta \leq \alpha} V_{i,\beta}$$

where  $x^{\alpha-\beta} V_{i,\beta}$  is identified with  $V_{i,\beta}$ . In particular,  $F_i = \bigoplus V_{i,\beta} \otimes R(-\beta)$ .

Let  $\mathcal{T}$  denote the Taylor resolution [Ta] of  $I$ . Then there is an exact sequence of the form

$$0 \longrightarrow \sum_{\beta < \alpha} \mathcal{T}_\beta \longrightarrow \mathcal{T}_\alpha \longrightarrow \mathcal{T}_\alpha / \sum_{\beta < \alpha} \mathcal{T}_\beta \longrightarrow 0,$$

and since the complex  $\mathcal{T}_\alpha$  is acyclic, this yields an isomorphism upon passing to homology

$$H_i(\mathcal{T}_\alpha / \sum_{\beta < \alpha} \mathcal{T}_\beta) \xrightarrow{\mu_i} H_{i-1}(\sum_{\beta < \alpha} \mathcal{T}_\beta) \cong \tilde{H}_{i-1}(\Theta(L_{<\alpha}), k)$$

for  $i \geq 2$ . This isomorphism is defined by

$$\mu_i([\bar{v}]) = [\partial^\mathcal{T}(v)]$$

whenever  $\bar{v}$  is a cycle in  $\mathcal{T}_\alpha / \sum_{\beta < \alpha} \mathcal{T}_\beta$  represented by the element  $v \in \mathcal{T}_\alpha$ . Observe that we have  $H_0(\sum_{\beta < \alpha} \mathcal{T}_\beta) = H_0(\Theta(L_\alpha), k)$  and hence also an isomorphism

$$H_1(\mathcal{T}_\alpha / \sum_{\beta < \alpha} \mathcal{T}_\beta) \xrightarrow{\mu_0} \tilde{H}_0(\Theta(L_{<\alpha}), k).$$

Let us make the following identifications in the minimal free resolution  $\mathcal{F}$  of  $I$ :

$$(F_\alpha/(\mathfrak{m}F)_\alpha)_i = (F_\alpha/\sum_{\beta \prec_B \alpha} F_\beta)_i = (F_\alpha)_i/\sum_{\beta \prec_B \alpha} (F_\beta)_i = \bigoplus_{\beta \leq_B \alpha} V_{i,\beta}/\bigoplus_{\beta <_B \alpha} V_{i,\beta} = V_{i,\alpha}.$$

Let us fix an embedding of  $\mathcal{F}$  into  $\mathcal{T}$ . Then  $\mathcal{T} = \mathcal{F} \oplus \mathcal{E}$  for some split exact complex of multigraded free modules  $\mathcal{E}$ .

**Proposition 5.2.** *Suppose  $I$  is Betti-linear. Fix  $\gamma \prec_B \alpha$ . Then for each  $i \geq 1$ , the diagram:*

$$\begin{array}{ccccc} V_{i,\alpha} & \xrightarrow{\partial} & (\sum_{\beta \prec_B \alpha} \mathcal{F}_\beta)_{i-1} & \xrightarrow{\text{proj}_\gamma} & V_{i-1,\gamma} \\ \text{incl} \downarrow & & & & \downarrow \text{incl} \\ Z_i(\mathcal{T}_\alpha/\sum_{\beta \prec_B \alpha} \mathcal{T}_\beta) & & & & Z_{i-1}(\mathcal{T}_\gamma/\sum_{\nu \prec_B \gamma} \mathcal{T}_\nu) \\ \text{proj} \downarrow & & & & \downarrow \text{proj} \\ H_i(\mathcal{T}_\alpha/\sum_{\beta \prec_B \alpha} \mathcal{T}_\beta) & & & & H_{i-1}(\mathcal{T}_\gamma/\sum_{\nu \prec_B \gamma} \mathcal{T}_\nu) \\ \mu_i \downarrow & & & & \downarrow \mu_{i-1} \\ H_{i-1}(\sum_{\beta \prec_B \alpha} \mathcal{T}_\beta) & & & & H_{i-2}(\sum_{\nu \prec_B \gamma} \mathcal{T}_\nu) \\ \parallel & & & & \parallel \\ \tilde{H}_{i-1}(\Theta(L_{<\alpha}), k) & & & & \tilde{H}_{i-2}(\Theta(L_{<\gamma}), k) \\ \iota_*^{-1} \downarrow & & & & \parallel \\ \tilde{H}_{i-1}(\Theta(B_{<\alpha}), k) & \longrightarrow & \tilde{H}_{i-2}(\Theta_{\alpha,\gamma}, k) & \longrightarrow & \tilde{H}_{i-2}(\Theta(L_{<\gamma}), k) \end{array}$$

is commutative.

*Proof.* Suppose that  $v \in V_{i,\alpha}$ . Note that the assumption of Betti-linearity yields

$$\partial^{\mathcal{F}}(v) = \sum_{\beta \prec_B \alpha} v_\beta$$

with each  $v_\beta \in V_{i-1,\beta}$ . Going down the left hand vertical side of the diagram we have  $\text{incl}(v) = v$ , and  $\text{proj}(v) = [v]$ . Following this,

$$\mu_i([v]) = [\partial^{\mathcal{T}}(v)] = [\partial^{\mathcal{F}}(v)] = [\sum_{\beta \prec_B \alpha} v_\beta].$$

Recall that  $\mu_i$  is an isomorphism for each  $i$ . The map  $\tilde{H}_{i-2}(\Theta(L_{<\alpha}), k) \rightarrow \tilde{H}_{i-2}(\Theta(B_{<\alpha}), k)$  sends  $[\sum_{\beta \prec_B \alpha} v_\beta]$  to the class  $[\sum_{\beta \prec_B \alpha} v_\beta + \partial\tau]$  for some boundary  $\partial\tau \in \Theta(L_{<\alpha})$ . Continuing along the bottom row, we write  $\tau = \tau' + \tau''$  where  $\tau'$  is a chain in  $\Theta(L_{\leq \alpha})$  and  $\tau''$  has no faces in  $\Theta(L_{\leq \alpha})$ . Then we have that  $\partial\tau = \partial\tau' + \partial\tau''$ , and

$$\partial_{i-1}^{\alpha,\gamma}([\sum_{\beta \prec_B \alpha} v_\beta + \partial\tau]) = [d_{i-2}(v_\gamma + d_{i-1}\tau'')]$$

where  $d_{i-2}$  is the usual simplicial boundary map. We see that

$$[d_{i-2}(v_\gamma + d_{i-1}(\tau''))] = [d_{i-2}(v_\gamma)] \in \tilde{H}_{i-3}(\Theta_{\alpha,\gamma}, k),$$

which is mapped to the same class considered in  $\tilde{H}_{i-3}(\Theta(L_{<\gamma}), k)$ .

Chasing the diagram in the other direction, we have that

$$\partial^{\mathcal{F}}(v) = \sum_{\beta \prec_B \alpha} v_{\beta}$$

hence

$$\text{proj}_{\gamma}(\sum_{\beta \prec_B \alpha} v_{\beta}) = v_{\gamma}$$

Of course, we know that  $\text{incl}(v_{\gamma}) = v_{\gamma}$ , and hence upon passing to homology we have  $[v_{\gamma}]$ . We then have that  $\mu_{i-1}([v_{\gamma}]) = [\partial^{\mathcal{T}}(v_{\gamma})]$ .

Finally, we must observe that  $[\partial^{\mathcal{T}}(v_{\gamma})] = [d_{i-2}(v_{\gamma})]$  in  $\tilde{H}_{i-3}(\Theta(L_{<\gamma}), k)$ .  $\square$

*Proof of Theorem 2.5:* We will show that the following diagram is commutative for each  $i \geq 1$ :

$$\begin{array}{ccc} \cdots \rightarrow \bigoplus \tilde{H}_{i-2}(\Delta(B_{<\lambda}), k) \otimes R(-\lambda) & \longrightarrow & \bigoplus \tilde{H}_{i-3}(\Delta(B_{<\lambda}), k) \otimes R(-\lambda) \rightarrow \cdots \\ \downarrow \Sigma \psi_i \otimes 1 & & \downarrow \Sigma \psi_{i-1} \otimes 1 \\ \cdots \rightarrow \bigoplus \tilde{H}_{i-2}(\Theta(B_{<\lambda}), k) \otimes R(-\lambda) & \longrightarrow & \bigoplus \tilde{H}_{i-3}(\Theta(B_{<\lambda}), k) \otimes R(-\lambda) \rightarrow \cdots \\ \downarrow \Sigma f_{i,\lambda} \otimes 1 & & \downarrow \Sigma f_{i-1,\lambda} \otimes 1 \\ \cdots \rightarrow \bigoplus V_{i,\lambda} \otimes R(-\lambda) & \longrightarrow & \bigoplus V_{i-1,\lambda} \otimes R(-\lambda) \rightarrow \cdots \end{array}$$

However, the top square of this diagram is commutative due to Proposition 4.6. The map  $f_{i,\lambda}$  is the following composition:

$$f_{i,\lambda} : \tilde{H}_{i-2}(\Theta(B_{<\lambda}), k) \longrightarrow \tilde{H}_{i-2}(\Theta(L_{<\lambda}), k) \longrightarrow H_i(\mathcal{T}_{\lambda} / \sum_{\beta \prec_B \lambda} \mathcal{T}_{\beta}) \longrightarrow V_{i,\lambda}$$

where all arrows involved are the inverses of the vertical maps from Proposition 5.2. Now, Proposition 5.2 immediately implies the commutativity of the bottom square. Since each  $f_{i,\lambda}$  and each  $\psi_i$  are isomorphisms, the result follows.  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY AT ALBANY, SUNY, ALBANY, NY 12222  
E-mail address: `dwood@math.albany.edu`